

TUW-00-02

# The virtual black hole in 2d quantum gravity

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## Abstract

As shown recently (W. Kummer, H. Liebl, D.V. Vassilevich, Nucl. Phys. B **544**, 403 (1999)) 2d quantum gravity theories — including spherically reduced Einstein-gravity — after an exact path integral of its geometric part can be treated perturbatively in the loops of (scalar) matter. Obviously the classical mechanism of black hole formation should be contained in the tree approximation of the theory. This is shown to be the case for the scattering of two scalars through an intermediate state which by its effective black hole mass is identified as a “virtual black hole”. The present discussion is restricted to minimally coupled scalars without and with mass. In the first case the probability amplitude diverges, except the black hole is “plugged” by a suitable boundary condition. For massive scalars a finite S-matrix element is obtained.

PACS numbers: 04.60.Kz, 04.70.Dy

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# 1 Introduction

Ever since the discovery of Black Hole (BH) evaporation [1] it has been evident that quantum processes involving a BH can exhibit quite unusual properties. In particular, it is not clear whether such a basic property of the  $S$ -matrix as unitarity can be preserved if BHs are present in intermediate states. At the present level of understanding of quantum general relativity it seems quite impossible to give a satisfactory description of such processes in the framework of full four-dimensional theory. Therefore, the study of simplified models is so important, among which spherically reduced Einstein gravity (SRG) is the physically most relevant one. Some important information can be collected already at the classical limit. Due to the progress in computer simulations the understanding of spherically symmetric collapse of classical matter towards a BH has reached a remarkable level [3]. That collapse is governed by a critical threshold which even allows a simplified discussion in terms of self-similar solutions [3, 4]. On the other hand, it has already been conjectured [5] that for (in  $d = 2$ ) minimally coupled matter (i.e. not properly restricted to its s-wave part) this critical behaviour is absent, i.e. a BH is produced by an arbitrarily small amount of matter.

One approach to obtain the quantum version of such processes is the use of Dirac quantization for suitably defined operators which describe the collapse of matter. A very popular model in this context is the study of thin spherical shells. Using the Kuchař decomposition [6], especially for null (lightlike) shells important progress has been made recently [7]. We should also mention some earlier papers [2] where the problem of physical states in dilaton gravity without matter was addressed. Although certain aspects of the correspondence between physical states and BHs could be clarified, the application of these results to the quantum  $S$  matrix for matter fields was not possible in the framework of the reduced phase space formalism.

From the point of view of usual quantum field theory the path integral approach seems the most natural one. There (properly defined)  $S$ -matrix elements directly determine the “physical observables”, which in the Dirac approach are — in the opinion of the present authors — not so easily to be extracted from the “Dirac observables”. During the last years the path integral quantization of general 2d gravity theories, including SRG coupled to matter, has shown considerable progress [8, 9]. Based upon the (global and local) dynamical equivalence of (torsionless) dilaton theories with first order 2d gravity in Cartan variables (with nonvanishing torsion) it turned out that

the 2d geometry can be integrated much more easily, if a specific gauge, the “light-cone” gauge for the Cartan variables [10] is chosen. The 2d metric corresponding to this gauge coincides with the Eddington-Finkelstein (EF) metric which has the advantage of avoiding coordinate singularities at an eventual horizon. As shown in [9] the explicit one loop contributions in the generating functional originate from the Gaussian integral of the scalars and from the “back reaction” due to the scalars through the covariant measure.

Our present work concentrates on the classical part of this functional, i.e. on the zero loop order (tree approximation) in terms of the scalar matter fields. Clearly here features similar to the ones in classical collapse can be expected, although those “macroscopic” effects could well be modified at the “microscopic” level, when we refer in the latter case, say, to the scattering of individual scalar quanta in 2d gravity theory. As will be shown, these effects are indeed present, emerging naturally *without any further ad hoc assumptions* from the 2d quantum gravity formalism.

In order to make this evident we shortly review the main arguments of the latter in section 2, generalizing it to the case of massive scalars. The classical vertex of scalar fields is extracted in section 3. The problems arising for the scattering amplitude from our restriction to minimally coupled massless and massive scalars are discussed in section 4. In section 5 we summarize our results.

## 2 2d quantum gravity with massive scalars

### 2.1 Path integral quantization

The total action  $L = L^{(g)} + L^{(m)}$  consists of a geometric part which is most conveniently written in first order form involving Cartan variables

$$L^{(g)} = \int_{M_2} [X^+ D e^- + X^- D e^+ + X d\omega + \epsilon \mathcal{V}(X^+ X^-, X)] \quad , \quad (1)$$

$$\mathcal{V} = X^+ X^- U(X) + V(X) \quad (2)$$

where  $De^a = de^a + (\omega \wedge e)^a$  is the torsion two form, the scalar curvature  $R$  is related to the spin connection  $\omega$  by  $-\frac{R}{2} = *d\omega$  and  $\epsilon$  denotes the volume two form  $\epsilon = \frac{1}{2}\epsilon_{ab}e^a \wedge e^b = d^2x \det e_\mu^a = d^2x e$ . Our conventions are determined by  $\eta = \text{diag}(1, -1)$  and  $\epsilon^{ab}$  by  $\epsilon^{01} = -\epsilon^{10} = 1$ . We also have to stress that even with holonomic indices,  $\epsilon^{\mu\nu}$  is always understood to be the antisymmetric

Levi-Civita symbol and never the corresponding tensor.  $L^{(g)}$  is globally and locally equivalent to the general dilaton theory

$$L_{(dil)} = \int d^2x \sqrt{-g} \left( -X \frac{R}{2} - V(X) + \frac{U(X)}{2} (\nabla X)^2 \right) , \quad (3)$$

determined by the *same* functions  $V$  and  $U$  of the dilaton field  $X$  as in (1) and (2). In (3)  $g_{\mu\nu}$  is the 2d metric,  $R$  the Ricci scalar. Spherically reduced gravity (SRG) is the special case  $U_{SRG} = -(2X)^{-1}$ ,  $V_{SRG} = -2$ .

The matter action we write directly in terms of components of the zweibeine  $e_\mu^a$  in  $e_\mu^a e_\nu^b \eta_{ab}$ , converting the usual expression for the Lagrangian with non-minimally coupled scalar fields  $S$  ( $\mathcal{F}(X) = X$  for SRG)<sup>1</sup>

$$\mathcal{L}^{(m)} = \frac{\mathcal{F}(X)}{2} \sqrt{-g} (g^{\mu\nu} \partial_\mu S \partial_\nu S - m^2 S^2) \quad (4)$$

into

$$\mathcal{L}^{(m)} = -\frac{\mathcal{F}(X)}{2} \left[ \frac{\varepsilon^{\alpha\mu} \varepsilon^{\beta\nu}}{e} \eta_{ab} e_\mu^a e_\nu^b \partial_\alpha S \partial_\beta S - m^2 S^2 e \right] . \quad (5)$$

In our paper, as in [9], we treat the simple case  $\mathcal{F} = 1$  of minimal coupling. This will be enough to see some of the basic features.

For the quantum theory — as well as for the much simplified treatment of the exact classical solutions to (1) or (3) — the use of the Eddington-Finkelstein (EF) gauge

$$e_0^+ = \omega_0 = 0, \quad e_0^- = 1 \quad (6)$$

has been found to be useful. It is convenient to introduce the shorthand notation for “coordinates”, “momenta” and related sources

$$\begin{aligned} q_i &= (\omega_1, e_1^-, e_1^+) , \\ p_i &= (X, X^+, X^-) , \\ j_i &= (j, j^+, j^-) , \\ J_i &= (J, J^-, J^+) . \end{aligned} \quad (7)$$

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<sup>1</sup>A further generalization, including selfcouplings is possible without difficulties. Those terms could also provide the necessary counterterms for the renormalization when quantum corrections to scalar vertices are included. Note, however, that such a self-interaction only gives rise to local contributions, while the matter-vertices derived by means of our effective theory are non-local in general (see below).

Following the canonical steps of constructing the path integral [9], after integrating out the auxiliary variables and the conjugate momentum to  $S$  for the gauge (6) the path integral reads

$$W = \int \sqrt{\det q_3} (\mathcal{D}S)(\mathcal{D}^3q)(\mathcal{D}^3p) \det F \cdot \exp i \int \left( \frac{\mathcal{L}_{(1)}^{\text{eff}}}{\hbar} + \mathcal{L}_{(s)} \right) d^2x \quad (8)$$

where the effective Lagrangian, derived from (3) and (5) becomes

$$\mathcal{L}_{(1)}^{\text{eff}} = -q_i \dot{p}_i + q_1 p_2 - q_3 \mathcal{V} - q_2 (\partial_0 S)^2 + (\partial_0 S)(\partial_1 S) - q_3 \frac{m^2}{2} S^2. \quad (9)$$

It is well known that the correct diffeomorphism invariant measure for a scalar field  $S$  on a curved background  $e_\mu^a$  is  $d((-g)^{1/4} S) = (d\sqrt{e} S)$ , where  $e = \det e_\mu^a$  [11]. Note, that  $\sqrt{-g} = e = e_1^+ = q_3$  in the EF gauge is a consequence of the gauge choice (6).  $\det F$  is the determinant resulting from the integration of the auxiliary variables of the extended Hamiltonian. It depends on the differential operator

$$F = \partial_0 + p_2 U(p_1) \quad (10)$$

The source term includes also the source  $Q$  for the scalar field as well as sources  $J_i$  for the momenta:

$$\mathcal{L}_{(s)} = j_i q_i + J_i p_i + S Q \quad (11)$$

Eq. (9) possesses the crucial property to be linear in the “coordinates”  $q_i$ . Also the factor  $\sqrt{\det q_3}$  in the measure may be either lifted into a Lagrangian type term by the integration over auxiliary ghost fields that can be integrated out later again, or by introduction of an auxiliary metric [9]. In both cases the linearity of the effective action is preserved. Thus integrating  $d^3q$  simply produces three  $\delta$ -functions

$$\delta \left( -\nabla_0 \left( p_1 - \hat{B}_1 \right) \right) \quad (12)$$

$$\delta \left( -\nabla_0 \left( p_2 - \hat{B}_2 \right) \right) \quad (13)$$

$$\delta \left( -F \left( p_3 - \hat{B}_3 \right) \right) \quad (14)$$

where  $F$  is defined in (10).  $\hat{B}_i$  are functions of the sources  $j_i$  and matter fields and will be given below. Using these three  $\delta$ -functions the integrations

over  $(d^3p)$  yield directly

$$p_i = \hat{B}_i \quad . \quad (15)$$

This simply means that in the phase-space (path-) integral only classical paths contribute to the  $p_i$  and the remaining continuous physical degrees of freedom are represented by the scalar field alone, since all integrations over geometric variables have been performed exactly. Note that integration over  $p_3$  from (14) produces another factor  $(\det F)^{-1}$  so that the total Faddeev-Popov determinant is one — a result consistent with experience from Yang-Mills fields in temporal gauges like (6).

By  $\nabla_0 = \partial_0 + i(\mu + i\varepsilon)$  we define a regularized time derivative with  $\varepsilon, \mu \rightarrow +0$  for describing the IR and UV regularized one-dimensional associated Green function in loop integrals. Homogeneous modes always appear when we invert the operator  $\nabla_0$ . Such modes ( $\nabla_0 \bar{p}_i = 0$ ) must be included in  $\hat{B}_i$  where they completely describe the (eventual) classical background [9]<sup>2</sup>.

$$\hat{B}_1 = \underbrace{\bar{p}_1 + \nabla_0^{-1} \bar{p}_2 + \hbar(\nabla_0^{-1} j_1 + \nabla_0^{-2} j_2)}_{:= B_1} - \nabla_0^{-2} (\partial_0 S)^2, \quad (16)$$

$$\hat{B}_2 = \underbrace{\bar{p}_2 + \hbar \nabla_0^{-1} j_2 - \nabla_0^{-1} (\partial_0 S)^2}_{:= B_2}, \quad (17)$$

$$\hat{B}_3 = e^{-\hat{T}} \left[ \nabla_0^{-1} e^{\hat{T}} (\hbar j_3 - V(\hat{B}_1) - \frac{m^2}{2} S^2) + \bar{p}_3 \right] \quad (18)$$

$$= \underbrace{e^{-T} [\nabla_0^{-1} e^T (\hbar j_3 - V(B_1)) + \bar{p}_3]}_{:= B_3} + \text{terms } \mathcal{O}(S^2). \quad (19)$$

$$\hat{F} = e^{-\hat{T}} \nabla_0 e^{\hat{T}}, \quad \hat{T} = \nabla_0^{-1} (\hat{U} \hat{B}_2), \quad \hat{U} = U(\hat{B}_1) \quad . \quad (20)$$

In the abbreviations an exponential representation for the operator  $F$  is used. There is still an ambiguity in the path integral. Indeed, the term  $\int J_3 \hat{B}_3$  can

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<sup>2</sup>E.g. for SRG that background may be a BH or flat Minkowski spacetime, depending on the choice of modes  $\bar{p}_i$ . The freedom still encoded in the homogeneous modes can be reduced by fixing the residual gauge freedom of the EF gauge (6) and solving Ward identities. This procedure is described in detail in [9].

be formally rewritten as

$$\int \left( e^{\bar{T}} (\hbar j_3 - V(\hat{B}_1) - \frac{m^2}{2} S^2) (-\nabla_0) e^{-\hat{T}} J_3 + J_3 \bar{p}_3 \right) \quad (21)$$

We have a freedom to add a homogeneous solution  $\nabla_0 \tilde{g} = 0$  to the term  $e^{-\hat{T}} J_3$ . This amounts to adding to the effective Lagrangian the term

$$\tilde{\mathcal{L}} = \tilde{g} e^{\hat{T}} \left( \hbar j_3 - \hat{V} - \frac{m^2}{2} S^2 \right). \quad (22)$$

The same procedure applied to the terms  $J_1 \hat{B}_1$  and  $J_2 \hat{B}_2$  just leads to trivial contributions. Clearly  $\tilde{\mathcal{L}}$  alone survives when the sources  $J_i$  for the momenta are switched off. Nevertheless, those sources are technically important for a simple definition of an overall conservation law  $d\mathcal{C} = 0$ , peculiar to all  $2d$  theories, even with interacting matter [12]. Its geometric part ( $Q = \int^{p_1} U(y) dy$ )

$$\mathcal{C}^{(g)} = e^{Q(p_1)} p_2 p_3 + \int^{p_1} V(u) e^{Q(u)} du \quad (23)$$

for SRG by fixing integration constants in a specific way may be defined as

$$\mathcal{C}_{SRG}^{(g)} = \frac{p_2 p_3}{\sqrt{p_1}} - 4\sqrt{p_1} \quad (24)$$

## 2.2 Effective scalar theory

Having performed the integral  $(\mathcal{D}^3 q)$  and then  $(\mathcal{D}^3 p)$  which is possible only in the chosen gauge (6) in a straightforward way, the generating functional (8) becomes [9]

$$W(j, J, Q) = \int (\mathcal{D}S) \exp i \int d^2 x \left( \frac{\mathcal{L}_{(2)}^{eff}}{\hbar} + SQ \right). \quad (25)$$

Scalar fields can be integrated only perturbatively. Let us separate different orders of  $S$  in the effective Lagrangian:

$$\mathcal{L}_{(2)}^{eff} = \mathcal{L}_0 + SQ + \mathcal{L}_2 + \mathcal{L}^{int} \quad (26)$$

where  $\mathcal{L}_0$  does not contain  $S$  and  $\mathcal{L}_2$  is quadratic in  $S$ . All higher powers are collected in  $\mathcal{L}^{int}$ . According to (25) the quadratic part  $\mathcal{L}_2$  describes a free minimally coupled scalar field on the effective background geometry with the zweibein expressed in terms of the external sources ( $T = \tilde{T}(S = 0)$ ):

$$\mathcal{L}_2 = (\partial_0 S) (\partial_1 S) - E_1^- (\partial_0 S)^2 - \frac{m^2}{2} S^2 E_1^+ \quad (27)$$

$$E_1^+ = e^T, \quad E_1^- = -\partial_0^{-2} \tilde{g} e^T (V' + UV) \quad (28)$$

We use capital letters  $E_1^\pm(J, j)$  to distinguish the effective values from the fundamental zweibein fields  $e_1^\pm$  which are already integrated out.

The interaction Lagrangian can be represented as

$$\mathcal{L}^{int}(S) \rightarrow \mathcal{L}^{int} \left( \frac{1}{i} \frac{\delta}{\delta Q} \right) \quad (29)$$

and pulled out from the integral over  $S$ . As shown in [9] the path integral measure for  $S$  by a straightforward redefinition can be reduced to just the standard Gaussian one. In the generating functional

$$\begin{aligned} W = & \exp \left\{ \frac{i}{\hbar} \int d^2x \mathcal{L}^{int} \right\} \\ & \times \exp \left\{ \frac{i\hbar}{2} \int_x \int_y Q(x) G_{xy} Q(y) + \int_x (J_i B_i) + i\Gamma^{1-loop}(j, J) \right\} \end{aligned} \quad (30)$$

where  $G_{xy}$  is the scalar field propagator on the effective background (28) and  $\Gamma^{1-loop}$  is the logarithm of the determinant which for  $m = 0$  may be expressed as a Polyakov action. We do not have to go into details on the one-loop contributions since the present paper deals with the tree-level diagrams which are determined by the first two terms in (30), where  $\mathcal{L}^{int}$  is to be interpreted as in (29).

### 3 Vertices of scalar field

In our present paper of primary interest are the effective scalar vertices contained in  $\mathcal{L}^{int}$ . At vanishing sources  $J_i = 0$  for the momenta,  $\mathcal{L}^{int}$  in (26) reduces to the corresponding contribution from  $\tilde{\mathcal{L}}$  as defined in (22).



### 3.1 Massless scalars

Let us first consider the case  $m = 0$ . Then the scalar field  $S$  enters the interaction Lagrangian only as  $(\partial_0 S)^2$ . Moreover, according to (15) and (16), which, in turn, are the input in (18) and (22), it always appears in the combination  $[\hbar j_2 - (\partial_0 S)^2]$ . Therefore, the effective vertex of order  $2n$  in  $\mathcal{L}^{int}$  of (26) has the generic form<sup>3</sup>:

$$S^{(2n)} = \int d^2 x_1 \dots d^2 x_n S^{(2n)}(x_1, \dots, x_n) (\partial_0 S)_{x_1}^2 \dots (\partial_0 S)_{x_n}^2 \quad (31)$$

where

$$S^{(2n)} = \frac{(-1)^n}{n!} \frac{\delta^n}{\delta j_2^n} \tilde{\mathcal{L}} \Big|_{j=0} = \frac{(-1)^{n-1}}{(n)!} \left( \frac{\delta^{n-1}}{\delta j_2^{n-1}} E_1^- \right) \Big|_{j=0} \quad (32)$$

with  $E_1^-$  defined in (28).

To obtain the  $(n-1)^{th}$  functional derivative of  $E_1^-$  it is enough to take  $j_2$  localized at  $n-1$  different points:

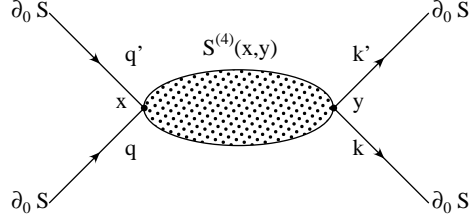
$$j_2(x) = \sum_{k=1}^{n-1} c_k \delta(y_k - x) \quad (33)$$

then we can expand  $E_1^-(j_2, x)$  in a power series of  $c_k$ . In the resulting sum the coefficient of the term with  $\prod_{k=1}^{n-1} c_k$  will give the desired functional derivative.

As seen from (20), (28)  $E_1^-$  is a nonlocal functional of the  $p_i$  which are again nonlocal functionals of the sources  $j_i$ . Our aim is the determination of the classical vertices. Therefore, their regularizations introduced in  $\nabla_0$  may be removed and  $\partial_0^{-1}$  simply becomes an integration  $\int dx_0$ . However, instead of applying this integration several times it is more convenient to solve the corresponding differential equations with suitable boundary conditions. This may seem surprising, because (32) in principle already represents the solution in closed form. But the treatment of multiple integrations is very involved with many, at first, undetermined integration ranges and integration constants, which — as for the BH in SRG — have singularities. Also the trick to go back to the classical equations which determine  $\tilde{\mathcal{L}}$  (cf. [9]) will give us important additional information to be used for the physical interpretation of the results.

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<sup>3</sup>From now on we put  $\hbar = 1$  for simplicity.

Figure 1:  $S^{(4)}$ -vertex with outer legs

Our starting point are the three differential equations for  $p_i$  which follow from solving the  $\delta$ -functions (12)-(14) (cf. eqs. (39-41) of [9]) in the presence of a source term  $j_2 q_2$  in  $\mathcal{L}$  whose solution is (15), in the special case  $S = 0, j_1 = j_3 = 0$ , i.e.  $(\partial_0 a = \dot{a})$ :

$$\dot{p}_1 - p_2 = 0 \quad (34)$$

$$\dot{p}_2 = j_2 \quad (35)$$

$$\dot{p}_3 + p_2 U p_3 + V = 0 \quad (36)$$

with  $j_2$  of (33). The quantity  $E_1^- = q_2(j_2)$  in the notation (7) may be calculated from the classical e.o.m.'s for  $q_i$  [9] or, equivalently, by suitable differentiations of (28) ( $U' = dU/dp_1$  etc.):

$$\dot{q}_1 - p_2 p_3 q_3 U' - q_3 V' = 0 \quad (37)$$

$$\dot{q}_2 + q_1 - q_3 p_3 U = 0 \quad (38)$$

$$\dot{q}_3 - p_2 q_3 U = 0 \quad (39)$$

From (38) and (39), eliminating  $\dot{q}_1$  by (37), or directly differentiating twice (28) the simple differential equations for  $q_2 = E_1^-$  may be obtained:

$$\ddot{q}_2 + q_3 (V' + UV) = 0 \quad (40)$$

where  $q_3 = E_1^+$  is already determined by the first Eq. (28).

In the following we restrict ourself to the vertex  $S^{(4)}(n=2)$  in (31). It is depicted in Fig. 1 (the momenta and corresponding arrows we included for later reference in the basic  $S$ -matrix elements with Minkowski modes). Then in (33) only one term with  $c_k = c$  is needed and

$$S^{(4)}(x, y) = -\frac{1}{2} \frac{\delta q_2(x)}{\delta j_2(y)}. \quad (41)$$

In  $S^{(4)}$  only the part of (41) contributes that is symmetric in  $x \leftrightarrow y$ . For  $E_1^-$  in (40) the only input from the momentum equation are the contributions to  $p_1$  and  $p_2$ . Fixing the residual gauge transformations these solutions with (33) at  $k = 1$  can be written as ( $x^0 = t, y^0 = s$ )

$$p_2 = 1 + c [\alpha + \Theta(t - s)] \quad (42)$$

$$p_1 = t + c [\alpha + \Theta(t - s)] (t - s) \quad (43)$$

where one integration constant has been absorbed in the definition of  $t$ . An overall factor  $\delta(x^1 - y^1)$  of the square brackets in (42) and (43), expressing the locality in  $x^1$ , is not written explicitly but will be taken into account in the end. The constant<sup>4</sup>  $\alpha$  parametrizes different possible solutions. There are three main cases:

- a)  $p_2 \neq 1$  for  $t > s$  only:  $\alpha = 0$
- b)  $p_2 \neq 1$  for  $t < s$  only:  $\alpha = -1$
- c) “symmetric” solution:  $\alpha = -\frac{1}{2}$ .

In the last case the square bracket in (42) and (43) may be replaced by half the sign function  $\varepsilon(t - s)$ . Although  $p_3$  does not enter eq. (40), it is necessary to compute the effective BH mass. From the general solution of (36)

$$p_3 = e^{-Q} \left( \bar{p}_3 - \int^{p_1} du V(u) e^{Q(u)} \right) \quad (44)$$

with  $\dot{\bar{p}}_3 = 0$  for each of the cases a) b) c) it is simple to find a solution with  $p_3^{t \leq s}(t = s) = p_3^{t \geq s}(t = s)$ . The most interesting application is SRG with  $U_{SRG} = -(2p_1)^{-1}$ ,  $V_{SRG} = -2$ . E.g. for case a) one finds easily with (34) for  $\mathcal{C}_{SRG}^{(g)}$

$$\mathcal{C}^{(g)}|_{t < s} = \bar{p}_3, \quad \mathcal{C}^{(g)}|_{t > s} = \bar{p}_3 + c (\bar{p}_3 + 4\sqrt{s}) \quad (45)$$

Here the integration constant  $\bar{p}_3$  must be independent of  $c$ . Thus in the term  $O(c)$ , relevant for our vertex  $S^{(4)}$  a nonvanishing effective “BH mass” has to be present. Since  $\mathcal{C}^{(g)} \propto -m_{BH}$  (cf. e.g. the last ref. [12]) with a “natural” choice  $\bar{p}_3 = 0$  for the solution without source  $j_2$ , a BH mass proportional

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<sup>4</sup>Here and in the following a ‘constant’ means also functions of  $x^1$  and  $s = y^0$ .

$(-\sqrt{4}s)$  will be switched on for  $t > s$ . As will be clarified below — despite our suggestive notation —  $t$  and  $s$  refer to a space coordinate. Our gauge choice (42) and (43) for SRG has placed the singularity at  $t = 0$  which, however, in this case would not lie in the region  $t > s$  where  $\mathcal{C}^{(g)}$  differs from zero. Thus case a) suggests the interpretation of a shell with negative mass, situated at  $t > s$ .

For case b) an analogous computation gives  $m_{ADM} \propto +4\sqrt{s}$  for  $t < s$  only, i.e. a proper BH at  $t = 0$ , whose effect is switched off for  $t \geq s$ . For c)  $\mathcal{C}^{(g)}$  jumps from  $-2\sqrt{s}$  at  $t < s$  to  $+2\sqrt{s}$  at  $t > s$ . The common feature of this apparently highly ambiguous situation (also other values of  $\alpha$  may be taken in (42) and (43)!) is the discontinuity in the effective BH mass at  $t = s$  which will make the appearance of the a singularity in  $S^{(4)}$  unavoidable. We call this phenomenon “virtual Black Hole” (VBH). Actually the ambiguity in  $\alpha$  *disappears altogether* in the vertex  $S^{(4)}(x, y) = S^{(4)}(y, x)$  so that the different interpretations in a), b), c) and for other values for  $\alpha$  should not be taken at face value. It should be noted that the range of variables  $t = x^0, s = y^0$  are not to be identified as the variables to be used in a scattering amplitude  $S + S \rightarrow S + S$  connecting asymptotic Minkowski space scalar fields (see below).

We now turn to the solution of (39) and (40), using case a) for (42) and (43) in anticipation of the fact that it will be symmetrized anyhow to the only relevant contribution to  $S^{(4)}$ . On the other hand, for another vertex to appear below for massive scalars, only that case will produce a finite result.

The continuous solution to (39) for SRG is (the indices  $(0)$  and  $(1)$  refer to  $t < s$  and  $t > s$ , respectively; the integration constant in  $Q$  is fixed according to  $Q_{SRG} = T_{SRG} = -\frac{1}{2} \ln t$ )

$$q_3^{(0)} = \frac{\bar{q}_3}{\sqrt{t}}, \quad (46)$$

$$q_3^{(1)} = \frac{\bar{q}_3}{\sqrt{t}} \left[ 1 - c \left( 1 - \frac{s}{t} \right) \right]. \quad (47)$$

Introducing (46), (47) into (40) for continuous  $q_2$  and  $\dot{q}_2$  at  $s = t$

$$q_2^{(0)} = \bar{q}_3 \left[ 4\sqrt{t} - \frac{2t}{\sqrt{s}} - 2\sqrt{s} + \bar{a}t + \bar{b} \right], \quad (48)$$

$$q_2^{(1)} = \bar{q}_3 \left[ \frac{4}{(1+c)^2} \sqrt{t(1+c) - sc} + \frac{2}{1+c} \left( \sqrt{s} - \frac{t}{\sqrt{s}} \right) - \frac{4\sqrt{s}}{(1+c)^2} + \bar{a}t + \bar{b} \right], \quad (49)$$

there is still a dependence on integration constants  $\bar{q}_3, \bar{a}, \bar{b}$ . It should be noted that  $\bar{q}_3$  in a direct calculation from  $\tilde{\mathcal{L}}$  in (21) would be replaced by the factor  $\tilde{g}(x^1)$ .

Next we will see that these constants can be fixed uniquely by natural assumptions for the effective line element, computed in the gauge (6) from  $q_2 = E_1^-, q_3 = E_1^+$  (we set  $x^1 = x$ ):

$$(ds)^2 = 2q_3(dt + q_2 dx)dx \quad (50)$$

For  $t < s$  with (46) and (48) in case a) we require the line element to describe flat (Minkowski) space, i.e. with a new coordinate  $\bar{t}$

$$(ds)_{(0)}^2 = 2d\bar{t}dx + (dx)^2 = (d\tau)^2 - (dz)^2 \quad (51)$$

This completely (only up to a sign in  $\bar{q}_3$  which we chose to be positive) fixes

$$b = sa = 2\sqrt{s}, \bar{q}_3 = \frac{1}{2\sqrt{2}}, \quad (52)$$

$$\sqrt{2}\bar{t} = \sqrt{t} \quad (53)$$

Otherwise a BH and an acceleration term (Rindler metric) would be present. In the last equality (51) the transition from (outgoing) EF coordinates to usual Minkowski coordinates

$$x = \tau - z, \quad \bar{t} = z \quad (54)$$

has been made. The relations (52) also lead to a unique result for the term of first order in  $c$  which determines

$$\frac{\delta q_2(x)}{\delta j_2(y)} = -\frac{1}{2\sqrt{2}} \frac{|\sqrt{x^0} - \sqrt{y^0}|^3}{\sqrt{x^0 y^0}} \delta(x^1 - y^1). \quad (55)$$

Here the symmetrization has been performed and the overall factor  $\delta(x^1 - y^1)$  included. This result is proportional to the (symmetrized) vertex calculated in [9], where the overall constant had not been determined.

It is also essential to study the effective line element  $(ds)_{(1)}^2$  valid in the range  $t \geq s$ . Again we may first bring it into EF form in terms of a new coordinate  $\bar{t}$ . Joining  $\bar{t}$  smoothly to the corresponding variable for  $t \leq s$  we get

$$\bar{t}_{(1)} = \frac{1}{\sqrt{2}} \left[ \sqrt{t} - \frac{1}{\sqrt{t}} + \sqrt{s} + c(\sqrt{s} - \sqrt{t}) \right] \quad (56)$$

which for large  $t$  and  $\bar{t}_{(1)}$  reduces to  $\bar{t}_{(1)} \rightarrow (\sqrt{t}(1 - c) + \sqrt{s}(1 + c)) / \sqrt{2}$ . In the line element

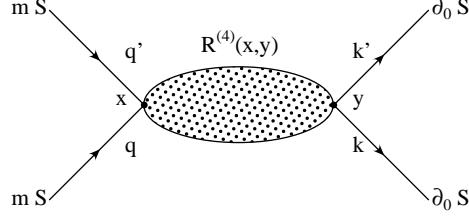
$$(ds)_{(1)}^2 = 2d\bar{t}_{(1)}dx + K_{(1)}(dx)^2 \quad (57)$$

the Killing norm  $K_{(1)}$  in terms of  $\bar{t}_{(1)}$  (for case a)) can be calculated easily to the required  $\mathcal{O}(c)$  from  $K_{(1)} = 2q_2^{(1)}q_3^{(1)}$  for asymptotic values of the radial variable  $\bar{t}_{(1)}$ :

$$\lim_{\bar{t}_{(1)} \rightarrow \infty} K = 1 + c \left[ \frac{3\sqrt{s}}{2\sqrt{2}\bar{t}_{(1)}} + \frac{\bar{t}_{(1)}}{\sqrt{2}s} - 3 + \mathcal{O}\left(\frac{1}{\bar{t}_{(1)}^2}\right) \right] \quad (58)$$

The first term in the square bracket, in agreement with (45), describes the VBH as a massive object with (negative) effective mass proportional to  $\sqrt{s}$ . Eq. (58) is valid asymptotically, but the linear dependence on  $\bar{t}_{(1)}$  indicates that it corresponds to a uniformly accelerated coordinate system with respect to Minkowski space. The acceleration is proportional to  $s^{-1/2} = (y_0)^{-1/2}$ . Thus in the simultaneous limit with  $y_0 \rightarrow \infty$  the asymptotic scalar fields, entering an  $S$ -matrix element to be computed from (55) in a certain sense may be determined by Minkowski modes after all.

It is obvious that similar arguments for case b) will produce flat Minkowski space for  $t > s$ . At  $t < s$  something like a genuine BH again together with linear terms in the radial variable  $\bar{t}_{(1)}$  appears. However, the restriction of the BH-like structure to the interval  $0 \leq t \leq s$ , at least for any finite  $s$  does not permit the definition of an asymptotic radial variable associated with some corresponding Rindler space. The “symmetric” case c) and all other situations with general  $\alpha$  in (42)-(43) have Rindler terms in the whole range of  $t$ . Thus the presence of an asymptotically flat Minkowski space on the sense of a double limit  $x^0 \rightarrow \infty, y^0 \rightarrow \infty$  (subjected to a very special sequence of those limits) is restricted to case a).

Figure 2:  $R^{(4)}$ -vertex with outer legs

### 3.2 Massive scalars

For massive scalars the last term in (22) again occurs in combination with a source for the zweibeine, namely  $j_3$ . For the vertex  $S^{(4)}$  now another term is created from an  $(\partial_0 S)^2$  in  $e^{\hat{T}}$  and that mass term. Therefore in the new vertex contribution

$$R^{(4)} = \int d^2x \int d^2y S_x^2 R^{(4)}(x, y) (\partial_0 S)_y^2 \quad (59)$$

we get ( $\hbar = 1$ )

$$R^{(4)} = -\frac{m^2}{2} \frac{\delta^2 \tilde{\mathcal{L}}}{\delta j_3(x) \delta j_2(y)} = -\frac{m^2}{2} \frac{\delta q_3(x)}{\delta j_2(y)} \quad (60)$$

This expression is simply the factor of  $c$  in (47). Together with the normalization of  $\bar{q}_3$  and the overall  $\delta(x^1 - y^1)$  we obtain in case a) of our different solutions for (42) and (43)

$$\frac{\delta q_3(x)}{\delta j_2(y)} = -\frac{\Theta(x^0 - y^0)}{2\sqrt{2}(x_0)^{3/2}}(x_0 - y_0). \quad (61)$$

The crucial difference to the other cases b) and c) consists in the property of (61) that only here the step function “protects” the  $x$  and  $y$  integrations from the singularity at  $x^0 = 0$  in  $R^{(4)}$  (see below).

## 4 Scattering Amplitude

For the scattering process of two scalars  $S + S \rightarrow S + S$  through the two vertex contributions  $S^{(4)} + R^{(4)}$  of the previous section we first transform

both  $x$  and  $y$  to the asymptotically flat coordinates (54). Clearly, in view of the remarks after (58), it may seem questionable that this is consistent with the properties of the vertex at asymptotic distances. Only in case a) an effective line element, say in  $x$ , exists which is asymptotically flat. But it refers to an accelerated system whose acceleration is proportional to  $(y_0)^{-1/2}$ . On the other hand, the free fields  $S(x)$  in the interaction picture approach to standard scattering theory cannot show a dependence on the variable  $y$  of a different vertex. This, in our opinion, justifies the assumption that (for case a)) there must be an asymptotic limit towards “independent” flat Minkowski space for  $S(x)$  and  $S(y)$  for a (hopefully) gauge fixing independent  $S$ -matrix element to exist. Then (31) with (41) and (55) yields  $(x^1 = \bar{x}^1)$  the manifestly nonlocal vertex

$$S^{(4)} = \frac{1}{64} \int d^2 \bar{x} \int d^2 \bar{y} \Theta(\bar{x}_0) \Theta(\bar{y}_0) \frac{(\partial_{\bar{x}_0} S)^2}{\bar{x}_0^2} \frac{(\partial_{\bar{y}_0} S)^2}{\bar{y}_0^2} |\bar{x}_0 - \bar{y}_0|^3 \delta(x_1 - y_1) \quad (62)$$

which is the same for  $m = 0$  and  $m \neq 0$  as well as for all possible  $\alpha$ -prescriptions.

In a similar manner the additional vertex for massive scalars from (6), (60) and (61) becomes

$$\begin{aligned} R^{(4)} &= -\frac{m^2}{8} \int d^2 \bar{x} \int d^2 \bar{y} \Theta(\bar{x}_0) \Theta(\bar{y}_0) \delta(x_1 - y_1) \\ &\quad \times \frac{\Theta(\bar{x}_0 - \bar{y}_0)(\bar{x}_0^2 - \bar{y}_0^2)}{\bar{x}_0^2 \bar{y}_0} S_{\bar{x}_0}^2 (\partial_{\bar{y}_0} S)^2. \end{aligned} \quad (63)$$

The vertex has been given for the case a) ( $\alpha = 0$ ) in (43).

## 4.1 Massless Scalars

Evidently both vertices (62) and (63) exhibit singularities at  $\bar{x}^0 = 0, \bar{y}^0 = 0$ . Let us consider asymptotic massless scalars first. Without imposing any boundary condition at  $\bar{x}^0 = 0$  the usual decomposition of the scalar field into Minkowski space modes will be

$$S = \frac{1}{\sqrt{2\pi}} \int \frac{dk}{\sqrt{2k}} \left( a_R^+ e^{ik(\tau-z)} + a_L^+ e^{ik(\tau+z)} + a_R^- e^{-ik(\tau-z)} + a_L^- e^{-ik(\tau+z)} \right), \quad (64)$$



where the indices  $R$  and  $L$  denote the right and left moving parts in our one-dimensional situation. In the “outgoing EF coordinates” (54) used in (6) the arguments are simply  $\tau - z = \bar{x}^1$ ,  $\tau + z = \bar{x}^1 + 2\bar{x}^0$ . The singularity of (65) in a scattering matrix element for two ingoing and outgoing Minkowski quanta of the  $S$ -field with momenta  $q, q'$  and  $k, k'$ , respectively

$$T(q, q'; k, k') = \frac{1}{2} \langle 0 | a^-(k) a^-(k') S^{(4)} a^+(q) a^+(q') | 0 \rangle \quad (65)$$

for any  $R$  and  $L$  can be interpreted as an indication that in this case the formation of a BH is “inevitable”. Indeed regularizing (62) with  $(\bar{x}^0)^2 \rightarrow \lim_{\delta \rightarrow 0} (\bar{x}^0{}^2 + \delta^2)^{-1}$  and defining a left moving wave packet formally by  $S \rightarrow \delta^{3/4} S$  would yield a finite result (up to an undetermined factor). The observed divergence of the scattering amplitude (65) for our vertex which contributes to the tree approximation thus seems to be in qualitative agreement with the conjecture [5], when scalar fields with minimal coupling are introduced at the reduced  $(1+1)$  level: Then the BH is formed for arbitrary small amounts of collapsing matter. A threshold for BH formation only occurs if nonminimally coupled scalars are considered, corresponding to proper taking into account of the  $s$ -wave nature in the spherically reduced situation [3].

A radical solution to the divergence problem consists in imposing a suitable boundary condition<sup>5</sup> for the scalar field to make (6) finite:

$$\left. \frac{\partial S}{\partial \bar{x}^0} \right|_{\bar{x}^0=0} = \left( \frac{\partial S}{\partial \tau} + \frac{\partial S}{\partial z} \right) = 0. \quad (66)$$

However, this eliminates the left-movers ( $a_L^\pm = 0$ ) in (62) also at  $\bar{x}^0 \neq 0$ , and (63), as well as in all higher order vertices. The physical system now consists of a background, eventually describing a fixed BH by a singularity at  $\bar{x}^0 = z = 0$ , and free right-moving scalars which run away from it. No genuine BH formation occurs in this setting.

## 4.2 Massive scalars

Although also in this case half of the modes are eliminated, a nontrivial result is obtained for free massive scalars with energy  $E_k = \sqrt{k^2 + m^2}$ , obeying the

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<sup>5</sup>This boundary condition implies that  $S$  becomes a self-dual scalar field at the origin - an essential difference to Dirichlet or Neumann boundary conditions, which - in a certain sense - are dual to each other [13].

boundary condition (66). The modes can be extracted from

$$S = \int_0^\infty dk N(k) \left\{ a^+(k) e^{iE(\bar{x}^0 + \bar{x}^1)} [(E_k + k) e^{-ik\bar{x}^0} - (E_k - k) e^{ik\bar{x}^0}] + h.c. \right\} \quad (67)$$

with the normalization factor

$$N(k) = [4\pi E_k (E_k^2 + k^2)]^{-1/2} \quad (68)$$

determined such that the Hamiltonian is  $H = \int_0^\infty dk E_k a^+ a^-$ .

Now

$$\frac{\partial S}{\partial \bar{x}_0} = 2m^2 \int_0^\infty dk N(k) \sin k\bar{x}_0 (a^+ e^{iE_k(\bar{x}_0 + \bar{x}_1)} + h.c.) \quad (69)$$

clearly obeys (66), but for  $m \neq 0$  it does not vanish identically any more. In the presence of boundary conditions,  $a^\pm(k)$  for positive, respectively negative energy in the S-matrix element (65) are related to values  $k \geq 0$ . Each  $k$  labels a mixture of left and right moving “particles”. With (69) the matrix element  $T^{(S)}$  of (65) is regular at  $\bar{x}^0 = \bar{y}^0 = 0$ . The same is true for the analogous one from  $R^{(4)}$ . In the latter the singularity  $\bar{x}_0^{-2}$  is absent *only* for the solution a) ( $\alpha = 0$  in (42) and (43)) thanks to the step functions. Both contributions to the total matrix element can be integrated completely yielding distributions. Some details of the calculation are described in Appendix A.

For “incoming” momenta  $q, q'$  with energies  $E_q, E_{q'}$  and outgoing ones  $k, k'$  with  $E_k, E_{k'}$ , taking into account symmetries and ensuing factors we obtain from (65)

$$\begin{aligned} T^{(S)}(q, q'; k, k') &= \frac{m^8}{64} N(q) N(q') N(k) N(k') \delta(E_k + E_{k'} - E_q - E_{q'}) \\ &\quad [I(q, q'; k, k') + I(q, -k; -q', k') + \\ &\quad + I(-k', q'; k, -q) + I(-k, -k'; -q, -q')] \end{aligned} \quad (70)$$

with

$$I := \sum_{abrs} abrs K_{ab,rs} \quad (71)$$

where the sum extends over  $a, b, r, s$ , each taking the values  $\pm 1$ . With the abbreviations ( $E = E_k + E'_k = E_q + E'_q$ )

$$\begin{aligned} u_{ab} &:= ak + bk' + E = \tilde{u}_{ab} + E \\ v_{rs} &:= rq + sq' + E = \tilde{v}_{rs} + E \end{aligned} \quad (72)$$

the quantity  $K_{ab,rs} := \lim_{\varepsilon \rightarrow 0} K(u_{ab}, v_{rs})$  is given by

$$\begin{aligned} K(u, v) &:= 2\pi i \frac{(u-v)^3}{u^2 v^2} \ln(u-v+i\varepsilon) \\ &\quad + \frac{\pi i}{v^2} (3v-u) \ln(u+i\varepsilon) \\ &\quad - \frac{\pi i}{u^2} (3u-v) \ln(-v-i\varepsilon). \end{aligned} \quad (73)$$

The logarithms of complex arguments are defined as usual, i.e.:

$$\lim_{\varepsilon \rightarrow 0} \ln(r \pm i\varepsilon) = \ln|r| \pm i\pi\Theta(-r) \quad (74)$$

The final result for the second contribution (63) inserted in (65) has a similar structure:

$$\begin{aligned} T^{(R)} &= \frac{m^6}{16} N(q)N(q')N(k)N(k')\delta(E_q + E'_q - E_k - E'_k) \\ &\quad [V(q, q'; k, k') + V(q, -k'; k, -q') + \\ &\quad + V(-k', q; k, -q) + V(-k, -k'; -q, -q')] \end{aligned} \quad (75)$$

$$V := V^{(1)} + V^{(2)} \quad (76)$$

$$V^{(1)} := - \sum_{abrs} rs(E_q + rq)(E'_q + sq')J_{abrs} \quad (77)$$

$$V^{(2)} := \sum_{abrs} rsb(E_q + rq)(E'_q + sq')L_{abrs} \quad (78)$$

In terms of  $\tilde{u}_{ab}$  and  $\tilde{v}_{rs}$  defined in (72) we have for  $J_{abrs} := \lim_{\varepsilon \rightarrow 0} J(\tilde{u}_{ab}, \tilde{v}_{rs})$ ,  $L_{abrs} := \lim_{\varepsilon \rightarrow 0} L(\tilde{u}_{ab}, \tilde{v}_{rs})$  the expressions

$$J(\tilde{u}, \tilde{v}) := \frac{2\pi i}{\tilde{v}} \ln(\tilde{u} + \tilde{v} + i\varepsilon) \quad (79)$$

$$L(\tilde{u}, \tilde{v}) := -2\pi i \frac{(E - \tilde{v})}{(E - \tilde{u})^2} \ln(E - \tilde{v} - i\varepsilon) \quad (80)$$

For both contributions the common overall sign has been fixed by the choice  $2\sqrt{2}\bar{q}_3 = +1$  in (52). Both terms also share the energy conservation factor. Therefore, only a probability per unit of time with factor  $\frac{1}{2\pi} \delta(E_k + E'_k - E_q - E'_q)$  is a well defined quantity. The situation with respect to momenta is different because of the nonlocality of the vertex. Thus we encounter a situation similar to scattering at a fixed external “potential” in ordinary quantum mechanics (or, equivalently, in  $D = 0 + 1$  dimensional QFT).

It should be noted that in the infrared limit both amplitudes are proportional to  $m$  and hence vanish for the massless case in agreement with the previous discussion. For large energies<sup>6</sup> the amplitudes decrease rapidly:

$$T^{(S)}|_{UV} \propto m^8 \frac{\ln E}{E^7}, \quad T^{(R)}|_{UV} \propto m^6 \frac{\ln E}{E^5} \quad (81)$$

## 5 Summary and Outlook

Two dimensional path integral quantum gravity can now be based upon a well-defined formalism which — in a very specific gauge — allows to separate the exact, almost trivial, quantum integral of the geometric variables from the loop-wise effects of the scalars. In our present work we considered the classical, tree-approximation, limit for minimally interacting scalar fields  $S$ , starting from the path integral formalism. This implies the appearance of effective (classical)  $2n$ -vertices of scalar fields ( $n \geq 2$ ). Those vertices are highly nontrivial, because they yield — through the natural appearance of classical background phenomena — mathematical structures which allow the interpretation that an intermediate “virtual” BH is involved. We have studied this for the geometric action as derived by spherically reducing Einstein gravity. The scalar matter field was assumed to be coupled minimally at the  $d = 2$  level. We also concentrated upon the simplest nontrivial vertex  $S^{(4)}$  with four scalar fields.

For the massless case we found that the resulting nonlocal matrix element for unrestricted left- and right-moving scalar fields diverges as  $\int_0^{z_f} \frac{dz}{z^2}$  at that point in space-time which can be identified with the “location” of a singularity. However, imposing a suitable boundary condition upon  $S$

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<sup>6</sup>Note that for very large energies our perturbation theory breaks down since the effects from the scalar field are not “small” anymore; therefore, the energy should lie in the range  $m \ll E \ll E_{\text{Planck}}$ .

completely eliminates the scalar excitation moving towards the singularity, whereas the ones moving away decouple from the theory: The manifold has been “plugged” at the place where an eventual BH may have been formed. We believe that this result at the particle level shows a qualitative relation to a conjecture for macroscopic BH formation: There for minimally coupled scalars a BH forms without any threshold [5]. The divergence of a probability amplitude or, alternatively, an amplitude which is finite only for a wave packet, properly rescaled to tend to zero width at  $z = 0$ , seems to imply the same phenomenon.

As an example for a system where finite amplitudes for minimally coupled scalars can be obtained we also studied massive scalars, where the necessary boundary condition no longer prevents BH effects. Both, the vertex from the massless case, and another new one, induced by the mass-term, yield finite results which can be even represented by (complicated) sums of directions (plus or minus) of momenta in terms of (simple) functions and distributions. Overall energy conservation holds in the process  $S + S \rightarrow S + S$ . Momenta are not conserved, in general. Here we note parallels to recent work of P. Hájíček [7] on massless, but non-minimally coupled thin spherical shells. He found no residual BH for a collapsing shell with Dirac quantization. Also in that work the phenomenon has been observed which we have called “virtual BH”, consisting in a certain sense of a black and a white hole.

The next task [14] is to take into account also the proper nonminimal coupling of scalars at the  $2d$  level. Superficially no essential basic changes for the vertices may be expected: On the one hand, the measure of the integral will change as  $dzd\tau \rightarrow z^2 dzd\tau$ , because  $z$  will become a radial variable. On the other hand the scalar field will be reduced to the one describing  $s$ -waves in  $d = 4$ , i.e.  $S \rightarrow S/z$ . But e.g. the threshold effect known for macroscopic studies [3] should show up. In that case a detailed comparison with Dirac quantization as treated in [7] will be possible.

Our formalism is general enough so that any other 2d gravity theory, produced e.g. by spherical reduction of generalized Einstein gravities in  $d = 4$ , can be covered as well.

Of course, also the study of higher loop orders in the scalar fields, based upon the one-loop determinant (Polyakov type action) in the path integral, as well as of higher loops involving the vertices discussed here, together with propagators of the scalars, remains a wide field of possible further applications.

## Acknowledgment

The authors thank their colleagues at the Institute for Theoretical Physics of the Vienna University of Technology for many discussions. This paper has been supported by project P-12815-TPH of the Austrian Science Foundation (FWF). One of the authors (D.V.) has been supported in part by the Alexander von Humboldt foundation and by RFBR, grant 97-01-01186.

## Appendix A: Derivation of Scattering Amplitudes

The explicit computation of the scattering amplitude  $T^{(S)}$  in (70) and  $T^{(R)}$  in (75) for Minkowski modes in the initial and final state is most conveniently based upon suitable Fourier transforms of the rational factors in  $\bar{x}^0$  and  $\bar{y}^0$  in (62) and (63). From the identity [15]

$$\int_0^\infty x^\lambda e^{i(\sigma+i\varepsilon)x} dx = ie^{\frac{i\lambda\pi}{2}} \Gamma(\lambda+1) (\sigma+i\varepsilon)^{-\lambda-1} \quad (82)$$

the required singular limits  $\delta \rightarrow +0, \varepsilon \rightarrow +0$  for the Fourier transforms at  $\lambda = -2 + \delta$ , resp.  $\lambda = -1 + \delta$  are

$$\int_{-\infty}^\infty x^{-2+\delta} \Theta(x) e^{i(\sigma+i\varepsilon)x} = f_2(\sigma, \varepsilon) \left[ \left( \frac{1}{\delta} + \left(1 + \frac{i\pi}{2} - \gamma\right) \right) - \ln(\sigma+i\varepsilon) + \mathcal{O}(\delta) \right] \quad (83)$$

resp.

$$\int_{-\infty}^\infty x^{-1+\delta} \Theta(x) e^{i(\sigma+i\varepsilon)x} = f_1(\sigma, \varepsilon) \left[ \left( \frac{1}{\delta} + \left(\frac{i\pi}{2} - \gamma\right) \right) - \ln(\sigma+i\varepsilon) + \mathcal{O}(\delta) \right], \quad (84)$$

with

$$f_n := i(-1)^{(n-1)} e^{-i\frac{n\pi}{2}} (\sigma+i\varepsilon)^{(n-1)} \quad (85)$$

Introducing (83), resp. (84) into (70) resp. (75) and using ( $P$  is Cauchy's

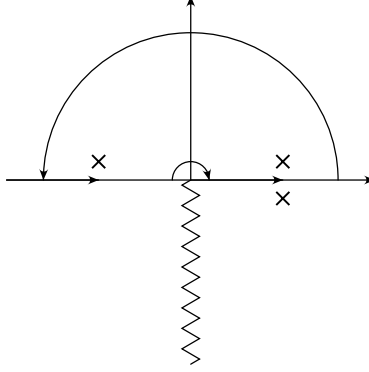


Figure 3: The complex contour typically encircles 2 singularities; note that for convenience we put the branch cut for the logarithm on the negative imaginary axis (depicted by the zigzag line).

principal value)

$$\begin{aligned}\varepsilon(x) &= \frac{1}{\pi i} P \int d\tau \frac{e^{i\tau x}}{\tau} \\ \Theta(x) &= \frac{1}{2\pi i} \int d\tau \frac{e^{i\tau x}}{\tau - i\varepsilon}\end{aligned}\tag{86}$$

all the terms proportional to  $\delta^{-1}$  from (83) and (84) cancel together with the constant contributions<sup>7</sup> — as they should in these finite integrals. Furthermore in (70) it is useful to replace the factor  $(\bar{x}^0 - \bar{y}^0)^3$  by a third derivative with respect to  $E = E_k + E'_k$ . Then the generic integral

$$A(a, b) = \int d\tau \ln(a + \tau + i\varepsilon_1) \ln(-b - \tau + i\varepsilon_2) (a + \tau)(b + \tau) \left( \frac{1}{\tau + i\varepsilon_3} + \frac{1}{\tau - i\varepsilon_3} \right)\tag{87}$$

remains which after three differentiations with respect to  $E$  in  $a = E + q + q'$ ,  $b = E + k + k'$  becomes a contribution of integrals with one logarithm multiplied by a factor with two or three poles. These integrals are straightforward and can be most conveniently done using the contour depicted in Fig. 3.

In the vertex  $T^{(R)}$  in (75) for the first contribution  $V^{(1)}$  the procedure is the

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<sup>7</sup>These cancellations are a direct consequence of the boundary condition (66).

same, not even requiring some differentiation at an intermediate step.  $V^{(2)}$  originates from the term with factor  $(-\bar{y}^0 \bar{x}^{-2})$ . Here  $\bar{y}^0$  may be expressed first by a derivative with respect to one of the momenta in the sine factor from  $(\partial_{\bar{y}} S)^2$  (cf. (69)).

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